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ELASTODYNAMIC STRESS INTENSITY FACTORS OF A CRACK NEAR AN INTERFACE

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Abstract—The elastodynamic problem of the diffraction of stress waves by a crack near an interface is considered. Using integral transform techniques the case of a crack perpendicular to the interface is reduced to a system of singular integral equations. The stress intensity factors at the crack tips are calculated by numerical techniques over a range of frequency for different material properties and crack distances from the interface.

1. INTRODUCTION

In recent years several papers have appeared dealing with the diffraction of waves for a finite crack. These papers deal with diffraction problems for either antiplane elasticity (slit crack), plane elasticity (slit crack), or three dimensional elasticity (penny-shaped crack). For all cases considered the problem is concerned with the impingement of a plane wave on an isolated crack in an infinitely extended homogeneous elastic body. To the author's knowledge composite media with finite cracks except for [1] have not been studied dynamically.

The purpose of the analysis herein is to dynamically analyze the stress field near a crack within a half plane which is bonded to a half plane of a different material. The crack is perpendicular to the bond plane, and an incident longitudinal wave is propagated through the system parallel to the crack. The mathematical discription is more complex than for antiplane problems because the scattered waves induced by the crack are composed of both longitudinal and transverse waves, even though the input wave may be of only one type, say a longitudinal wave. Results will be given in the form of stress intensity factors for several values of the shear moduli and the Poisson's ratios.

The problems are formulated by using integral transform techniques, and they are reduced to the solution of a system of singular integral equations. The method most suitable for carrying through the numerical analysis is the one of Erdogan and Gupta[2, 3], which is essentially a collocation scheme.

2. FORMULATION OF PROBLEM

An unbounded medium consists of two joined elastic half planes of different materials as shown in Fig. 1. The rectangular coordinate system (x, y, z), is oriented such that the plane $y = 0$ coincides with the interface. The material constants of the upper and lower half planes: shear modulus, Poisson's ratio and the mass densities, are denoted by μ_1, ν_1, ρ_1 and μ_2, ν_2, ρ_2

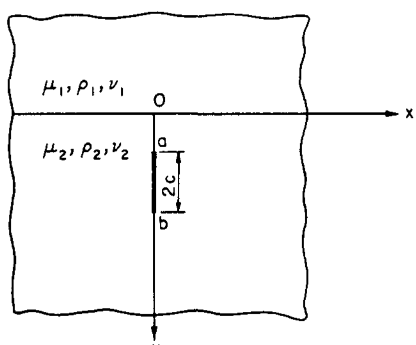

Fig. 1. Bonded half planes with a crack.

respectively. In the subsequent sections the subscripts (1) and (2), will refer to the upper and lower half plane. Let a vertical crack of length $2c$ be located in the lower half plane such that the crack is in the plane $x = 0$, and the tips of the crack are at distances a and b from the x-axis. The components of the displacements in the x, y, and z directions are given by u_x, u_y and u_z. For the plane strain problem, $u_z = 0$, and all derivatives with respect to z vanish. The equations of motion are

$$c_{Lj}^2\frac{\partial^2(u_x)_j}{\partial x^2}+c_{Tj}^2\frac{\partial^2(u_x)_j}{\partial y^2}+(c_{Lj}^2-c_{Tj}^2)\frac{\partial^2(u_y)_j}{\partial x\partial y}=\frac{\partial^2(u_x)_j}{\partial t^2} \tag{1}$$

$$c_{Lj}^2\frac{\partial^2(u_y)_j}{\partial y^2}+c_{Tj}^2\frac{\partial^2(u_y)_j}{\partial x^2}+(c_{Lj}^2-c_{Tj}^2)\frac{\partial^2(u_x)_j}{\partial x\partial y}=\frac{\partial^2(u_y)_j}{\partial t^2} \tag{2}$$

where $j = 1$ for $y \le 0$, $j = 2$ for $y \ge 0$ and where c_{Lj} are the velocities of waves of dilatation, and c_{Tj} are the velocities of waves of distortion, i.e.,

$$c_{Lj}^2 = (\lambda_j + 2\mu_j)/\rho_j,\ c_{Tj}^2 = \mu_j/\rho_j,\ j = 1, 2.$$

It is assumed that the composite medium is subjected to a plane longitudinal wave originating at $y = -\infty$, and normally incident to the interface of the composite medium. The incident displacement waves can be expressed in the form

$$u^{(o)} = A_0 e^{ik_0(y-c_{L1}t)} \tag{3}$$

where A_0 and k_0 are the amplitude and wave number of the incident waves. The problem of reflection and refraction of the incident wave has been discussed in the book by Achenbach [4].

For the case of normal incidence, the shear stresses are equal to zero and the normal stress $(\sigma_x)_2^{(o)}$ in the lower half plane can be expressed in the form

$$\begin{aligned}(\sigma_x)_2^{(o)} &= i\lambda_2 k_0 A_0 (c_{L1}/c_{L2})[2\rho_1 c_{L1}/(\rho_1 c_{L1} + \rho_2 c_{L2})]\\ &\quad\times \exp[i(k_0 c_{L1} y/c_{L2} - \omega t)].\end{aligned} \tag{4}$$

The expression given by equation (4) represents the stress in the lower half plane in the absence of a crack. To solve the diffraction problem the composite with a crack must be analyzed, where the crack surfaces have a distribution of normal tractions equal and opposite to those given by equation (4). The displacement solutions to the latter problem are the scattered waves and they are denoted by $(u_x)_1^{(s)}(x, y, t)$, $(u_y)_1^{(s)}(x, y, t)$, $(u_x)_2^{(s)}(x, y, t)$ and $(u_y)_2^{(s)}(x, y, t)$. The complete solution will be

$$(u_x)_j = (u_x)_j^{(o)} + (u_x)_j^{(s)} \tag{5}$$

$$j = 1, 2$$

$$(u_y)_j = (u_y)_j^{(o)} + (u_y)_j^{(s)}. \tag{6}$$

Here, the superscripts o and s refer to the incident and scattered waves.

Assuming harmonic motion of the form $(u_x, u_y, 0)e^{-i\omega t}$ equations (1) and (2) reduce to

$$\nabla^2\epsilon^* + (k_1^*)^2\epsilon^* = 0, \quad y \le 0 \tag{7}$$

$$\nabla^2\omega_z^* + (k_2^*)^2\omega_z^* = 0, \quad y \le 0 \tag{8}$$

and identical equations for ϵ, ω_z involving k_1, k_2 for $y \ge 0$ where ϵ^*, ϵ and ω_z^*, ω_z are the dilatation and the z component of rotation, respectively, and

$$k_1^* = \omega/c_{L1} = k_0,\ k_2^* = \omega/c_{T1},\ k_1 = \omega/c_{L2},\ k_2 = \omega/c_{T2}.$$

If the crack surfaces are to be free from tractions, it is necessary to require that

$$(\sigma_x)_2^{(o)} + (\sigma_x)_2^{(s)} = 0,\ x = 0,\ a < y < b. \tag{9}$$

Since the loading of the crack surfaces implies u_y, τ_{xx}, τ_{yy} are symmetric in x, the boundary conditions may be written on $x = 0$ as follows:

$$(\tau_{xy})_1^{(s)} = 0 \quad -\infty < y < 0 \tag{10}$$

$$(\tau_{xy})_2^{(s)} = 0 \quad 0 < y < \infty \tag{11}$$

$$(u_x)_2^{(s)} = 0 \quad 0 < y < a, b < y < \infty \tag{12}$$

$$(u_x)_1^{(s)} = 0 \quad -\infty < y < 0. \tag{13}$$

The continuity conditions of $y = 0$, $x \geq 0$ are given by

$$(\sigma_y)_1^{(s)} = (\sigma_y)_2^{(s)} \tag{14}$$

$$(\tau_{xy})_1^{(s)} = (\tau_{xy})_2^{(s)} \tag{15}$$

$$[(u_x)_1^{(s)}]' = [(u_x)_2^{(s)}]' \tag{16}$$

$$[(u_y)_1^{(s)}]' = [(u_y)_2^{(s)}]' \tag{17}$$

where $[\ ]' = \mathrm{d}[\ ]/\mathrm{d}x$.

The solutions of equations (7) and (8) for the problem under consideration may be obtained in the form of Fourier transforms and the displacements appropriate to each region given as follows:

For $y \leq 0$

$$(u_x)_1^{(s)}(x, y) = \frac{2}{\pi}\int_0^\infty [\xi(k_1^*)^{-2}F\mathrm{e}^{\alpha_1^* y} + 2(k_2^*)^{-2}\alpha_2^* G\mathrm{e}^{\alpha_2^* y}]\,.\,\sin(\xi x)\,\mathrm{d}\xi \tag{18}$$

$$(u_y)_1^{(s)}(x, y) = -\frac{2}{\pi}\int_0^\infty [\alpha_1^*(k_1^*)^{-2}F\mathrm{e}^{\alpha_1^* y} + 2\xi(k_2^*)^{-2}G\mathrm{e}^{\alpha_2^* y}]\cos(\xi x)\,\mathrm{d}\xi. \tag{19}$$

For $y \geq 0$

$$\begin{aligned}(u_x)_2^{(s)}(x, y) &= \frac{2}{\pi}\int_0^\infty [\xi k_1^{-2}A\mathrm{e}^{-\alpha_1 y} - 2k_2^{-2}\alpha_2 C\mathrm{e}^{-\alpha_2 y}]\sin(\xi x)\,\mathrm{d}\xi \\ &\quad + \frac{2}{\pi}\int_0^\infty [\alpha_1 k_1^{-2}B\mathrm{e}^{-\alpha_1 x} + 2k_2^{-2}\xi D\mathrm{e}^{-\alpha_2 x}]\cos(\xi y)\,\mathrm{d}\xi\end{aligned} \tag{20}$$

$$\begin{aligned}(u_y)_2^{(s)}(x, y) &= \frac{2}{\pi}\int_0^\infty [\alpha_1 k_1^{-2}A\mathrm{e}^{-\alpha_1 y} - 2k_2^{-2}\xi C\mathrm{e}^{-\alpha_2 y}]\cos(\xi x)\,\mathrm{d}\xi \\ &\quad + \frac{2}{\pi}\int_0^\infty [\xi k_1^{-2}B\mathrm{e}^{-\alpha_1 x} + 2k_2^{-2}\alpha_2 D\mathrm{e}^{-\alpha_2 x}]\sin(\xi y)\,\mathrm{d}\xi\end{aligned} \tag{21}$$

where

$$\begin{aligned}\alpha_j^* &= [\xi^2 - (k_j^*)^2]^{1/2} = -i[(k_j^*)^2 - \xi^2]^{1/2} \\ \alpha_j &= (\xi^2 - k_j^2)^{1/2} = -i(k_j^2 - \xi^2)^{1/2} \quad j = 1, 2.\end{aligned} \tag{22}$$

The stresses that correspond to equations (18)–(21) can be easily computed by Hooke's law and will not be given here. Equations (18)–(21) will give outgoing waves for the scattered wave solution provided that the convention given in equation (22) is used. From equations (18)–(21) it can be seen that there are six Fourier transforms to be solved from the boundary and continuity conditions (9)–(17). Equations (10) and (13) are automatically satisfied by symmetry.

Equation (11) leads to the algebraic relation

$$B(\xi) = -(\alpha_2^2 + \xi^2)\beta_2^2 D(\xi)/\xi\alpha_1 \tag{23}$$

where

$$\beta_1^2 = (1-2\nu_1)/2(1-\nu_1),\ \beta_2^2 = (1-2\nu_2)/2(1-\nu_2). \tag{24}$$

Equation (12) can be automatically satisfied by defining $D(\xi)$ in the following manner:

$$D(\xi) = \int_a^b b(s)\sin(\xi s)\,\mathrm{d}s \tag{25}$$

where $b(s)$ has the physical meaning of dislocation density. Equations (15) and (17) lead to algebraic relations from which, F and G may be given in terms of A and C. Equations (14) and (16) lead to two integral relations. By taking the Fourier cosine transforms of these relations and observing that the resulting infinite integrals have integrands that can be expanded in the form of partial fractions, one can obtain equations for A and C in a relatively simple form. The dislocation density, $b(s)$, can be deduced from the remaining boundary condition, equation (9). By using techniques mentioned above for the Fourier transforms, equation (9) can be reduced to the singular integral equation given next,

$$\int_a^b \frac{b(s)}{s-y}\,\mathrm{d}s + \int_a^b b(s)\left[\frac{1}{s+y} + K(y,s)\right]\mathrm{d}s = (-\pi/2\mu_2)[k_2^2/(k_1^2-k_2^2)](\sigma_x)_2^{(o)},$$
$$a < y < b,\ x = 0 \tag{26}$$

where

$$K(y,s) = (k_1^2-k_2^2)^{-1}\int_0^\infty \left\{\left[\frac{(2\xi^2-k_2^2)^2-4\alpha_1\alpha_2\xi^2}{\xi\alpha_1} - 2(k_1^2-k_2^2).\right]\sin(\xi s)\cos(\xi y)\right.$$
$$\left. + \alpha_1^*(k_2^*)^2\frac{[f_1(\xi,y)(A_1+B_1\mathrm{e}^{-\alpha_1 s}+C_1\mathrm{e}^{-\alpha_2 s}) + f_2(\xi,y)(A_1^*+B_1^*\mathrm{e}^{-\alpha_1 s}+C_1^*\mathrm{e}^{-\alpha_2 s})]}{E_1(\xi)E_2(\xi)+P_1(\xi)P_2(\xi)}\right\}\mathrm{d}\xi \tag{27}$$

and

$$A_1^* = -(k_1k_2/\alpha_1)^2,\ B_1^* = 2\xi^2 + k_2^2[k_2^2-(k_1\alpha_2/\alpha_1)^2]/(k_2^2-k_1^2),\ C_1^* = -2\xi^2 \tag{28}$$

$$A_1 = (k_2/\alpha_1)^2(k_2^2-2k_1^2),\ B_1 = 4\xi^2-(k_2/\alpha_1)^2(k_2^2-2k_1^2),\ C_1 = -4\xi^2 \tag{29}$$

$$f_1(\xi,y) = (\mu_2/\mu_1)\{[2\xi^2+(k_2^2-2k_1^2)]E_1(\xi)\mathrm{e}^{-\alpha_1 y} + 2\xi\alpha_2P_2(\xi)\mathrm{e}^{-\alpha_2 y} \tag{30}$$

$$f_2(\xi,y) = -[2\xi^2+(k_2^2-2k_1^2)]\xi^{-1}P_1(\xi)\mathrm{e}^{-\alpha_1 y} + 2\alpha_2E_2(\xi)\mathrm{e}^{-\alpha_2 y} \tag{31}$$

$$E_1(\xi) = (k_2^*)^2(\xi^2+\alpha_1^*\alpha_2) + [2\xi^2-(\mu_2/\mu_1)(\alpha_2^2+\xi^2)][\alpha_1^*\alpha_2^*-\xi^2] \tag{32}$$

$$E_2(\xi) = (k_2^*)^2\{\alpha_1[2\xi^2-(k_2^*)^2] + \alpha_1^*(\mu_2/\mu_1)(2\xi^2-k_2^2)\}$$
$$+ 2(\mu_2/\mu_1-1)\alpha_1\xi^2[2\xi^2-(k_2^*)^2-2\alpha_1^*\alpha_2^*] \tag{33}$$

$$P_1(\xi) = \xi\{(k_2^*)^2[2\xi^2-(k_2^*)^2+2\alpha_1^*\alpha_2\mu_2/\mu_1]$$
$$+ [2\xi^2-(\mu_2/\mu_1)(\xi^2+\alpha_2^2)][2\alpha_1^*\alpha_2^*-2\xi^2+(k_2^*)^2]\} \tag{34}$$

$$P_2(\xi) = \xi\{-(k_1^*)^2[\alpha_1^*+\alpha_1] + 2(\mu_2/\mu_1-1)\alpha_1(\alpha_1^*\alpha_2^*-\xi^2)\}. \tag{35}$$

In solving the singular integral equation (26), the consistency condition

$$\int_a^b b(s)\mathrm{d}s = 0 \tag{36}$$

must also be applied.

The diffracted waves may give rise to Stonely waves which are not diminished at a large distance of x, and the kernel, $K(y,s)$, will always have one Cauchy type singularity. If this is the case, the kernel may be evaluated in the sense of its principal value. For the values of material properties to be discussed in the next section of Numerical Analysis and Conclusions this phenomena will not occur.

3. NUMERICAL ANALYSIS AND CONCLUSIONS

The numerical analysis is concerned with the solution of equation (26) subject to condition (36). The analysis will be separated into the two cases: (a) crack within the lower half plane ($a > 0$) and (b) crack extending to the interface of the composite ($a = 0$). The reasons for such a separation are the same as discussed in [1] when the layer was fully cracked. In the analysis to follow the densities of the two materials, ρ_1, ρ_2 will be taken equal for convenience.

(a) *Case for* $a > 0$. It is convenient to introduce the nondimensional variables as follows:

$s = c\tau + (a + c) = c\tau_1$, $y = cy' + (a + c) = cs_1$, $\xi = k_0\eta$, $k_1^* = k_0 = \beta/c$, $\delta_1 = (\mu_1\beta_2^2/\mu_2\beta_1^2)^{1/2}$, $\delta_2 = (\mu_1/\mu_2\beta_1^2)^{1/2}$, $\delta_3 = 1/\beta_1$ and $\varphi(\tau) = b(\tau)/k_0A_0$, where β_1 and β_2 are given in equations (24). Noting that the unknown function, $\varphi(\tau)$, and kernel, $K(s_1, \tau_1)$ are complex, the integral equation (26) can be written as two coupled singular integral equations whose kernel and unknown functions are real and are given by

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi_1(\tau)\mathrm{d}\tau}{\tau - s} + \frac{1}{\pi}\int_{-1}^{1} \left[\left(K_1 + \frac{1}{\tau_1 + s_1}\right)\varphi_1 - K_2\varphi_2\right]\mathrm{d}\tau = m_0 \sin(\beta\delta_1 s_1) \tag{37}$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi_2(\tau)}{\tau - s}\mathrm{d}\tau + \frac{1}{\pi}\int_{-1}^{1} \left[K_2\varphi_1 + \left(K_1 + \frac{1}{\tau_1 + s_1}\right)\varphi_2\right]\mathrm{d}\tau = -m_0 \cos(\beta\delta_1 s_1) \tag{38}$$

where

$$\varphi(\tau) = \varphi_1(\tau) + i\varphi_2(\tau),\ K(s_1, \tau_1) = K_1(s_1, \tau_1) + iK_2(s_1, \tau_1)$$

and

$$m_0 = (\lambda_2/\mu_2)\{\delta_1^2/[(\beta_2^2 - 1)(\delta_1 + 1)]\}. \tag{39}$$

The singular integral equations (37) and (38) are to be solved with the following subsidiary conditions:

$$\int_{-1}^{1} \varphi_1(\tau)\mathrm{d}\tau = 0 \tag{40a}$$

$$\int_{-1}^{1} \varphi_2(\tau)\mathrm{d}\tau = 0. \tag{40b}$$

The solution of equations (37) and (38) with subsidiary conditions, equations (40) can be carried out by means of the Gauss–Chebyshev integration formula in the manner devised by Erdogan and Gupta [2].

By expressing the solution of (37) and (38) as

$$\varphi_1(\tau) = G_1(\tau)/(1 - \tau^2)^{1/2}, \quad j = 1, 2 \tag{41}$$

and using the Gauss–Chebyshev integration formula to evaluate the integrals in these equations, a system of $2n - 2$ algebraic equations for the $2n$ unknowns is obtained. Remaining equations are obtained from equation (40).

After evaluating G_1 and G_2, the stress intensity factors K_a and K_b for the crack tips $y = a$, b, respectively, may be obtained as

$$K_a = \lim_{y \to a} (a - y)^{1/2}(\sigma_y)_2^s(0, y) \tag{42a}$$

$$K_b = \lim_{y \to b} (y - b)^{1/2}(\sigma_y)_2^s(0, y). \tag{42b}$$

Equations (42) lead to:

$$K_a/\mu_2 k_0 A_0 c^{1/2} = 2(\delta_2^2 - \delta_1^2)/\delta_2^2 \,.\, G(-1) \tag{43a}$$

$$K_b/\mu_2 k_0 A_0 c^{1/2} = -2(\delta_2^2 - \delta_1^2)/\delta_2^2 \,.\, G(1). \tag{43b}$$

Note that $(K_a)_1$, $(K_b)_1$ and $(K_a)_2$, $(K_b)_2$ correspond to the real and imaginary part of the stress intensity factors K_a and K_b, respectively. The results will be given in terms of $|K_a|$ and $|K_b|$.

(b) *Case for* $a = 0$. When the upper crack tips reaches the interface, $a = 0$, its singularity is no longer square root, but will have a power dependence.

Taking the limit as $\omega \to 0$, equation (26) reduces to the static problem in which the crack is opened by a constant internal pressure as in [5]. The nature of singularity for the static case has been studied in detail by Barnett [6] and thus is omitted here. It is easy to show that the forms for $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are given by

$$\varphi_1(\tau) = G_1(\tau)(1+\tau)^{\gamma-1}(1-\tau)^{-1/2} \tag{44a}$$

$$\varphi_2(\tau) = G_2(\tau)(1+\tau)^{\gamma-1}(1-\tau)^{-1/2} \tag{44b}$$

where γ is a constant determined from the equation

$$\cos(\pi\gamma) = \alpha_3 - \beta_3\gamma^2 \quad 0 < \gamma \le 1 \tag{45}$$

where

$$\alpha_3 = \tfrac{1}{2}(A_3 + B_3), \quad \beta_3 = 2A_3 \tag{46}$$

and

$$A_3 = (1 - \mu_1/\mu_2)/[1 + (3 - 4\nu_2)\mu_1/\mu_2] \tag{47}$$

$$B_3 = [(3 - 4\nu_1) - (3 - 4\nu_2)\mu_1/\mu_2]/[\mu_1/\mu_2 + (3 - 4\nu_1)]. \tag{48}$$

The solution can be carried out by a procedure similar to the one given above but using the Gauss–Jacobi integration formula [3].

The "stress intensity factors" K_a and K_b may be obtained from the equation

$$K_a = \lim_{y \to a} (a - y)^{-(\gamma-1)}(\sigma_y)_2^s(0, y)$$

or

$$K_a/\mu_2 k_0 A_0 c^{1-\gamma} = 2(\delta_2^2 - \delta_1^2)/\delta_2^2 G(-1) \tag{49}$$

and K_b remains the same as given in (43b).

Numerical results were computed for normalized stress intensity factor versus normalized wave number for a range of material and geometrical parameters. One set of curves is shown in Fig. 2 to show qualitative behavior. The curves show that for relatively low wave-numbers, $\beta < 0{\cdot}5$, the stress intensity factors at end a may be greater than that at end b. For $\beta > 0{\cdot}7$ the stress intensity factors seem to follow the same trends for ends a and b with the stress intensity factor for end b *always* being greater than end a.

To compare the results of the composite problem with that of the homogeneous medium, numerical results were taken for the case, $\mu_1/\mu_2 = 0{\cdot}95$, which is nearly homogeneous. The results are plotted in Fig. 3. Their results show the clear tendency of the composite problem to the homogeneous problem as $\mu_1/\mu_2 \to 1$. It is noted that the stress intensity factor of the static case must be multiplied by a factor m when comparing with the dynamic case where

$$m = \lambda_2\delta_2^2 \,.\, 2\delta_1\mu_2(\delta_1^2 - \delta_2^2). \tag{50}$$

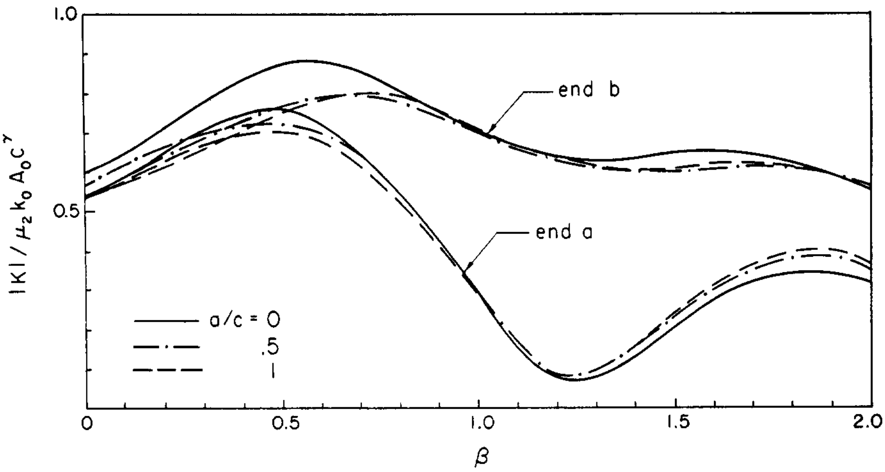

Fig. 2. Stress-intensity factor vs wave-number for $\nu_1 = \nu_2 = 0{\cdot}25$, $\mu_1/\mu_2 = 0{\cdot}5$ and various values of a/c ($\gamma = 0{\cdot}575$ for $a/c = 0$: $\gamma = 1/2$ otherwise.)

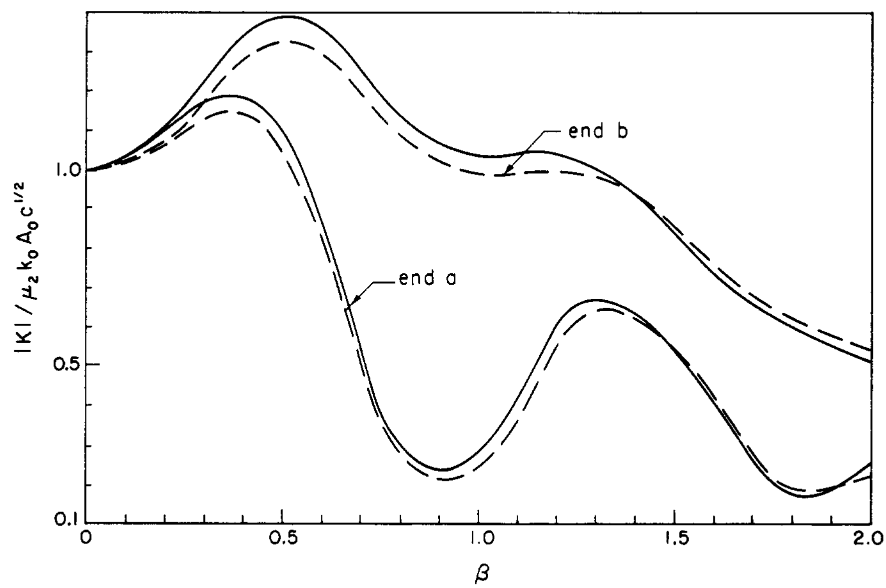

Fig. 3. Stress-intensity factor vs wave-number for the comparison of composite medium with homogeneous medium ($\nu_1 = \nu_2 = \nu = 0{\cdot}25$, $\mu_1/\mu_2 = 0{\cdot}95$, $a/c = 0{\cdot}5$: dot lines are results of composite medium).

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REFERENCES

1. L. M. Keer and W. C. Luong, Diffraction of waves and stress intensity factors in a cracked layered composite. *Acoust. Soc. America.* **56**, 1681 (1974).
2. F. Erdogan and G. D. Gupta, On the numerical solution of singular integral equations. *Quar. Appl. Math.* **29**, 525 (1972).
3. F. Erdogan, G. D. Gupta and T. S. Cook, Numerical solution of singular equations. Published in *Methods of Analysis and*
4. J. D. Achenbach, *Wave Propagation in Elastic Solids.* North-Holland, American Elsevier (1973).
5. J. G. Kuang and T. Mura, Dislocation pile-up in two-phase materials. *J. Appl. Phys.* **39**, 109 (1968).
6. D. M. Barnett, Edge dislocation stress concentrations in two-phase media. *Scripta Metallurgica.* **3**, 543 (1969).